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A GENERAL VARIATIONAL THEOREM FOR THE RATE PROBLEM IN ELASTO-PLASTICITY

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Abstract—An elastic–plastic strain-hardening solid in a quasi-static state of finite deformation is considered. A new, general variational principle is established for the boundary-value problem in which incremental displacements are prescribed on one portion of the boundary surface of a body and increments of nominal traction are prescribed on the remaining portion. Both Eulerian and Lagrangian formulations of this general variational principle are given. It is shown that certain extremum principles due to Hill [11, 12] and Murphy and Lee [20] may be considered as special cases of the newly established variational theorem. It is also illustrated how this variational principle may be applied to obtain an approximate solution for the problem of elastic–plastic torsion of circular bars with variable diameter.

1. INTRODUCTION

IN A series of well-known papers [1–4], Reissner has presented a general variational theorem for stresses and displacements which provides the starting point for the derivation of specialized extremum principles in elasticity. The classical principles of minimum potential and maximum complementary energies, for example, may be considered as special cases of his general theorem.

Reissner's theorem is such that both equilibrium equations and stress–strain relations are Euler equations of the variational problem while boundary conditions for stresses as well as displacements are natural (Euler) boundary conditions. In view of the direct methods of the calculus of variations, Reissner's principle thus provides a useful basis which permits independent selection of both the stress distributions and displacement fields in the approximate solution of specific boundary-value problems in elasticity [5, 6]. Reissner has also shown that his general variational principle can be applied to establish the appropriate field equations for the theory of plates [1, 2] and shells [7], which represents a two-dimensional approximation of a three-dimensional theory.

Typical problems in plasticity often involve changes in geometry that cannot be neglected. The analysis of such problems generally requires the formulation of the proper incremental or "rate problem", for infinitesimal deformation superimposed on finite deformation. This boundary-value problem involving rates of stress, velocities and prescribed rates of surface traction can then be solved in an incremental manner to furnish a step-wise solution to the equilibrium behaviour of the body. For rigid-plastic solids, rate formulations have been employed by Onat [8], Onat and Shu [9] and Batterman [10]. More general, elastic–plastic bodies have been considered by Hill [11, 12] who also

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established a uniqueness criterion for the solution of the rate problem and developed extremum principles characterizing the unique solution.

In this paper, a general variational principle for the rate boundary-value problem is presented for elastic–plastic strain-hardening solids. The theorem is in terms of stress rates (or increments) and velocities and is such that the rate equations of equilibrium as well as the constitutive relations are Euler (differential) equations of the variational principle and all boundary conditions for traction rates and velocities are the natural boundary conditions. Consequently, this variational theorem for finite deformations of elastic–plastic bodies represents a generalization of Reissner's principle in classical elasticity. Both Eulerian and Lagrangian formulations of the newly established variational theorem are given—the latter formulation often being more convenient for the solution of certain boundary-value problems in solid mechanics.

As an alternative formulation of the rate problem, the proposed variational principle has distinct advantages. It may be applied in conjunction with an incremental Ritz technique for the approximate solution of given elastic–plastic boundary-value problems involving finite deformations; or the theorem may be used to generate the proper systems of field equations for certain one- and two-dimensional approximations to the general theory. As an example of application, it is shown how the variational theorem may be applied to the problem of elastic–plastic torsion of circular bars of variable diameter.

2. EULERIAN FORMULATION OF THE RATE PROBLEM

For the case of finite deformations a distinction must be made between the original and deformed configurations of the continuum. Consequently two descriptions may be employed to define the relevant variables and to formulate the fundamental field equations [13, 14]—the Lagrangian or the Eulerian description referring to the original (undeformed) or the current (deformed) configuration of the body, respectively. Since constitutive relations for elastic–plastic solids are generally expressed in terms of a "true" stress tensor and rate-of-deformation tensor [11, 12, 15, 16], Eulerian variables are employed in the following formulation of the rate problem.

A general curvilinear coordinate system θ^i is taken as the reference frame. Before deformation a typical particle of the body has initial (Lagrangian) coordinates a^i with respect to the coordinate system θ^i. In its current state of deformation, the (Eulerian) coordinates of the particle with respect to the same reference system are denoted by χ^i. Subsequent deformation of the body from its current configuration can be described by the rate-of-deformation tensor and rate-of-rotation tensor given, respectively, by

$$\dot{\varepsilon}_{ij} = \tfrac{1}{2}[v_{i,j} + v_{j,i}] \tag{1a}$$

and

$$\dot{\omega}_{ij} = \tfrac{1}{2}[v_{i,j} - v_{j,i}] \tag{1b}$$

where v_i are the covariant components of the velocity vector and the comma denotes covariant differentiation with respect to the coordinates χ^i.

The equations of equilibrium for the current stress distribution are

$$\sigma^{ij}_{\ \ ,i} + \rho f^j = 0 \tag{2}$$

in which σ^{ij} represents the contravariant components of the true (Eulerian) stress tensor, ρ is the density of the material in its current configuration and f^j are the components of the specific body force. In rate form the equations for continuing (incremental) equilibrium become

$$\dot{\sigma}^{ij}_{,i}+\sigma^{ij}_{,i}v^k_{,k}-\sigma^{jk}_{,i}v^i_{,k}+\rho\dot{f}^j = 0 \tag{3}$$

where the dot denotes the material derivative [13]; i.e. the rate of change following the element.

For the development of the proposed variational theorem, it is convenient to express the rate equation of equilibrium (3) as follows [11, 12]

$$\dot{s}^{ij}_{,i}+\rho\dot{f}^j = 0 \tag{4}$$

where

$$\dot{s}^{ij} = \dot{\sigma}^{ij}+\sigma^{ij}v^k_{,k}-\sigma^{jk}v^i_{,k} \tag{5}$$

represents a "nominal" stress-rate tensor (unsymmetrical) based on the current configuration such that the rate of nominal traction on an element of area dS of the deformed body is given by

$$\dot{F}^j = n_i\dot{s}^{ij} \tag{6}$$

in which n_i are the components of the unit normal of the element in its current (deformed) configuration. Consequently, the rate of loading on this element of area becomes

$$\mathrm{d}\dot{P}^j = \dot{F}^j\,\mathrm{d}S = n_i\dot{s}^{ij}\,\mathrm{d}S. \tag{7}$$

For the current configuration of equilibrium, it is supposed that the stress distribution, state of deformation and material parameters have already been determined. In the theory of elastic–plastic solids, the typical boundary-value problem consists of determining the increments or rates of change of these quantities when the velocity field is prescribed on part of the current boundary surface S_v, the nominal traction rates are prescribed on the remainder S_F and the body force rates are prescribed in the volume V. Thus the boundary conditions of the rate problem are

$$v_j = v_j^* \quad \text{on } S_v \tag{8}$$

$$\dot{F}^j = n_i\dot{s}^{ij} = \dot{F}^{j*} \quad \text{on } S_F \tag{9}$$

where an asterisk denotes a prescribed quantity.

3. CONSTITUTIVE RELATIONS

The constitutive law for an elastic–plastic solid is generally expressed as a relation between the rate-of-deformation tensor and a stress-rate tensor. An objective stress rate which vanishes under rigid-body rotation must be employed. The Jaumann derivative [13] of the Kirchhoff stress tensor based on the current configuration, given by [12]

$$\frac{D\tau^{ij}}{Dt} = \frac{D\sigma^{ij}}{Dt}+\sigma^{ij}v^k_{,k} \tag{10a}$$

where

$$\frac{D\sigma^{ij}}{Dt} = \dot{\sigma}^{ij} - \sigma^{ik}\dot{\omega}_k^j - \sigma^{kj}\dot{\omega}_k^i \tag{10b}$$

satisfies this requirement since it is associated with axes rotating with the material element but not deforming with it. Since the vanishing of this stress rate also implies that the invariants of the true stress tensor and deviatoric stress components are stationary [15, 16], it is preferable to employ Jaumann's definition of stress rate in the constitutive equations of plasticity.

The classical strain-hardening elastic–plastic solid, as generalized by Hill [11, 12] has the following constitutive law

$$\frac{D\tau^{ij}}{Dt} = K^{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^p) \tag{11a}$$

where

$$\dot{\varepsilon}_{ij}^p = \begin{cases} \frac{1}{h} m_{ij} m_{kl} \frac{D\tau^{kl}}{Dt} & \text{when } m_{ij}\frac{D\tau^{ij}}{Dt} > 0 \\ 0 & \ldots \leq 0. \end{cases} \tag{11b}$$

In these relations K_{ijkl} are elastic moduli (symmetric with respect to all indices), $\dot{\varepsilon}_{ij}^p$ represents the plastic part of the rate-of-deformation tensor, h is a positive scalar measure of the current rate of hardening and m_{ij} are components of the outward normal to the current yield surface in six-dimensional stress space. From (11a) and (11b) the constitutive relation becomes

$$\frac{D\tau^{ij}}{Dt} = K^{ijkl}\dot{\varepsilon}_{kl} - \begin{cases} K^{ijkl} m_{kl} \dfrac{m_{pq} K^{pqrs}\dot{\varepsilon}_{rs}}{h + m_{pq}K^{pqrs}m_{rs}} & \text{when } m_{ij}\dfrac{D\tau^{ij}}{Dt} > 0 \\ 0 & \ldots \leq 0. \end{cases} \tag{12}$$

This constitutive law can also be expressed in the form [11]

$$\dot{s}^{ij} = \frac{\partial E(v)}{\partial(v_{j,i})} \tag{13}$$

where

$$\begin{aligned} E(v) &= \frac{1}{2}\dot{s}^{ij} v_{j,i} \\ &= \frac{1}{2}\frac{D\tau^{ij}}{Dt}\dot{\varepsilon}_{ij} - \sigma^{ij}\dot{\varepsilon}_i^k\dot{\varepsilon}_{kj} + \frac{1}{2}\sigma^{ij}v_{,i}^k v_{k,j} \end{aligned} \tag{14}$$

is expressed as a homogeneous potential function of degree two in the velocity gradients by means of (14b) and the constitutive equation (12). The (unique) inverse of (13) becomes

$$v_{j,i} = \frac{\partial E(\dot{s})}{\partial \dot{s}^{ij}} \tag{15}$$

where the potential $E(\dot{s})$ is expressed in terms of nominal stress-rate components through the Legendre transformation

$$E(\dot{s})+E(v) = \dot{s}^{ij}v_{j,i}. \tag{16}$$

This particular form of the constitutive relation also implies that certain other stress rates and strain rates (e.g. $\dot{\varepsilon}^{ij}$ and $D\tau^{ij}/Dt$) are derivable from potentials [12]. However, although $\dot{s}^{ij}$ is non-symmetric, it is convenient to employ this stress rate in order to facilitate the initial formulation of the proposed general variational principle. In view of the relations (5), (10) and (14) it is evident that subsequent formulations of the theorem will be obtainable in terms of the symmetric stress rates $\dot{\sigma}^{ij}$ or $D\tau^{ij}/Dt$.

4. GENERAL VARIATIONAL THEOREM

The analysis of finite deformations of elastic–plastic solids usually consists of trying to obtain the states of stress and strain in a given body as functions of a prescribed load and deformation history. For some relatively simple boundary-value problems, such as bodies in homogeneous states of stress, these relationships are readily obtained by solving the equations of equilibrium (2) in conjunction with the constitutive relations and boundary conditions. However, to determine the deformation behaviour for more complex elastic–plastic boundary-value problems one may resort to an incremental solution of the following system of rate equations

$$\dot{s}^{ij}_{,i}+\rho \dot{f}^{j} = 0 \quad \text{in } V \tag{17}$$

$$v_{j,i} = \frac{\partial E(\dot{s})}{\partial \dot{s}^{ij}} \quad \text{or } \dot{s}^{ij} = \frac{\partial E(v)}{\partial (v_{j,i})} \quad \text{in } V \tag{18}$$

subject to the prescribed boundary conditions

$$v_j = v_j^* \quad \text{on } S_v \tag{19}$$

$$\dot{F}^j = n_i\dot{s}^{ij} = \dot{F}^{j*} \quad \text{on } S_F. \tag{20}$$

The formulation of these equations in terms of the nominal stress-rate tensor $\dot{s}^{ij}$ has certain advantages. In addition to providing a concise form for the theory which takes finite deformations into account, the equations are analogous in form to the field equations of classical elasticity or to those characterizing a hyperelastic solid at finite deformation [14]. This immediately suggests that an equivalent variational formulation, similar to Reissner's theorem in elasticity, may be obtained for the rate problem. Consequently, guided by Reissner's principle we construct the following functional

$$I = \int_V [\dot{s}^{ij}v_{j,i}-\rho \dot{f}^{j}v_j - E(\dot{s})]\,\mathrm{d}V - \int_{S_F} \dot{F}^{j*}v_j\,\mathrm{d}S - \int_{S_v} \dot{F}^j(v_j-v_j^*)\,\mathrm{d}S \tag{21}$$

and state the following theorem: the variational problem which has the differential equations (17) and (18) as Euler equations and the boundary conditions (19) and (20) as natural boundary conditions is the problem

$$\delta I = 0. \tag{22}$$

For arbitrary variations of $\dot{s}^{ij}$ and v_j, $\delta\dot{F}^j = n_i\,\delta\dot{s}^{ij}$ is also arbitrary and the variation of I becomes

$$\delta I = \int_V [\dot{s}^{ij}\,\delta(v_{j,i}) + v_{j,i}\,\delta\dot{s}^{ij} - \rho\dot{f}^j\,\delta v_j - \delta E(\dot{s})]\,\mathrm{d}V - \int_{S_F} \dot{F}^{j*}\,\delta v_j\,\mathrm{d}S$$
$$- \int_{S_v} \dot{F}^j\,\delta v_j\,\mathrm{d}S - \int_{S_v} \delta\dot{F}^j(v_j - v_j^*)\,\mathrm{d}S.$$

From the relations

$$\delta(v_{j,i}) = (\delta v_j)_{,i} \qquad \delta E(\dot{s}) = \frac{\partial E(\dot{s})}{\partial \dot{s}^{ij}}\,\delta\dot{s}^{ij}$$

and the following transformation due to the divergence theorem of Gauss

$$\int_V \dot{s}^{ij}(\delta v_j)_{,i}\,\mathrm{d}V = -\int_V \dot{s}^{ij}_{,i}\,\delta v_j\,\mathrm{d}V + \int_S \dot{F}^j\,\delta v_j\,\mathrm{d}S \tag{23}$$

the above expression becomes

$$\delta I = \int_V \left[\left(v_{j,i} - \frac{\partial E(\dot{s})}{\partial \dot{s}^{ij}}\right)\delta\dot{s}^{ij} - (\dot{s}^{ij}_{,i} + \rho\dot{f}^j)\,\delta v_j\right]\mathrm{d}V$$
$$+ \int_{S_F} (\dot{F}^j - \dot{F}^{j*})\,\delta v_j\,\mathrm{d}S - \int_{S_v} \delta\dot{F}^j(v_j - v_j^*)\,\mathrm{d}S \tag{24}$$

which shows that the Euler equations of the variational problem are equations (17)–(20). Conversely, the solution of the rate problem is such that I is stationary in the class of continuous stress-rate and velocity fields.

In view of the Legendre transformation (16) equation (21) can be written as

$$I = \int_V [E(v) - \rho\dot{f}^j v_j]\,\mathrm{d}V - \int_{S_F} \dot{F}^{j*} v_j\,\mathrm{d}S - \int_{S_v} \dot{F}^j(v_j - v_j^*)\,\mathrm{d}S \tag{25}$$

whereas a transformation similar to (23) gives the following expression for I

$$I = -\int_V [(\dot{s}^{ij}_{,i} + \rho\dot{f}^j)v_j + E(\dot{s})]\,\mathrm{d}V + \int_{S_F} (\dot{F}^j - \dot{F}^{j*})v_j\,\mathrm{d}S + \int_{S_v} \dot{F}^j v_j^*\,\mathrm{d}S. \tag{26}$$

If the admissible velocity fields are restricted to take the prescribed values (19) on S_v, then from (25)

$$I \equiv J(v) = \int_V [E(v) - \rho\dot{f}^j v_j]\,\mathrm{d}V - \int_{S_F} \dot{F}^{j*} v_j\,\mathrm{d}S. \tag{27}$$

Similarly, for stress-rate distributions satisfying equilibrium (17) in V and the prescribed boundary conditions (20) on S_F, equation (26) gives

$$I \equiv G(\dot{s}) = \int_{S_v} \dot{F}^j v_j^*\,\mathrm{d}S - \int_V E(\dot{s})\,\mathrm{d}V. \tag{28}$$

Hill [12] has shown that when the solution to the (rate) boundary-value problem is unique, the functional $J(v)$ is an absolute minimum and $G(\dot{s})$ is an absolute maximum for the actual solution. These extremum principles, which are analogous to the potential energy and complementary energy principles in elasticity for an added state of stress and strain, can thus be considered to be consequences of the general variational theorem. They also indicate that any kinematically admissible velocity field satisfying (19) furnishes an upper bound to the functional I, whereas any statically admissible stress-rate distribution [satisfying equations (17) and (20)] furnishes a lower bound.

The relationships between the various stress rates [e.g. equations (5), (10) and (14)] can now be employed to furnish alternative expressions of the general variational theorem and corresponding extremum principles. For example, the functional (25) can be written as

$$I = \int_V [H(\dot{\varepsilon}) - \sigma^{ij}\dot{\varepsilon}_i^k\dot{\varepsilon}_{kj} + \tfrac{1}{2}\sigma^{ij}v^k_{,i}v_{k,j} - \rho\dot{f}^j v_j]\,\mathrm{d}V$$
$$- \int_{S_F} \dot{F}^{j*} v_j\,\mathrm{d}S - \int_{S_v} \dot{F}^j(v_j - v_j^*)\,\mathrm{d}S \tag{29}$$

where the potential

$$H(\dot{\varepsilon}) = \frac{1}{2}\frac{D\tau^{ij}}{Dt}\dot{\varepsilon}_{ij} \tag{30}$$

is expressed as a homogeneous function of degree two in the strain-rate components by means of the constitutive law (12).

5. APPLICATION OF THE THEOREM

To illustrate an application of the theorem, consider the problem of a circular bar of variable diameter fixed at one end and twisted axially by a couple applied at the other end. The bar has the form of a body of revolution, the axis of the bar coincides with the z-axis and polar coordinates r and θ are used for defining the position of an element in the plane of a cross-section. The end $z = 0$ of the bar is fixed, the lateral surface is free from external forces while the end $z = c$ is free from normal stress and subjected to a prescribed twisting couple M^*. Although elastic solutions are available [17] for a number of specific contours, there appears to be no specific contour for which the complete analytic elastic–plastic solution is known. As a result, numerical techniques have been employed to obtain approximate solutions to the problems of elastic–plastic torsion of circular bars of variable diameter [18].

An analysis for an isotropic elastic material [17] indicates that during twist, the particles move only in tangential directions such that transverse sections of the bar remain plane and circular, but radii of an undeformed transverse section become curved after deformation. The only non-zero stresses in this case are $\sigma_{\theta z}$, $\sigma_{r\theta}$ and because of symmetry all variables are independent of the angle θ. As an approximation for an isotropic elastic–plastic solid, it is now assumed that the same deformation mode can be used to describe each increment of elastic–plastic deformation such that for each current configuration

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{rz} = 0 \tag{31}$$

$$v_r = v_z = 0, v_\theta = v(r, z) \tag{32}$$

where physical components [14] of the stress tensor and velocity vector have been employed.

As a result of these assumptions and equations (5), (6) the functional corresponding to (21) becomes, for each current configuration

$$I = \int\int\left[\dot{\sigma}_{r\theta}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\dot{\sigma}_{\theta z}\frac{\partial v}{\partial z}-E(\dot{s})\right]r\,\mathrm{d}r\,\mathrm{d}z-\int_{z=c}\dot{\sigma}^{*}_{\theta z}vr\,\mathrm{d}r+\int_{z=0}\dot{\sigma}_{\theta z}vr\,\mathrm{d}r \tag{33}$$

where

$$\dot{M}^{*} = \int\int_{z=c} r\dot{F}^{*}_{\theta}\,\mathrm{d}S = 2\pi\int_{z=c} r^{2}\dot{\sigma}^{*}_{\theta z}\,\mathrm{d}r \tag{34}$$

and the potential function $E(\dot{s})$ is expressed in terms of stress-rate components through the particular constitutive law.

For an isotropic solid characterized by linear elastic behaviour combined with isotropic-hardening and a von-Mises dissipative potential, the inverse form of the constitutive relation (11) becomes, in terms of physical components

$$\dot{\varepsilon}_{ij} = \frac{1+\nu}{E}\frac{D\tau_{ij}}{Dt}-\frac{\nu}{E}\delta_{ij}\frac{D\tau_{kk}}{Dt}+\frac{\alpha}{h}m_{ij}m_{kl}\frac{D\tau_{kl}}{Dt} \tag{35a}$$

where

$$\alpha = \begin{cases} 1 \text{ when } m_{ij}\dfrac{D\tau_{ij}}{Dt} > 0 \\ 0 \quad \ldots \quad \leq 0 \end{cases} \tag{35b}$$

and

$$m_{ij} = \sigma'_{ij} = \sigma_{ij}-\tfrac{1}{3}\delta_{ij}\sigma_{kk}. \tag{35c}$$

In these expressions, E and ν are, respectively, the modulus of elasticity and Poisson's ratio, δ_{ij} denotes the Kronecker delta and σ'_{ij} is the deviatoric stress tensor. Consequently, the potential function to be used in (33) is in this case given by

$$E(\dot{s}) = \tfrac{1}{2}\dot{s}_{ij}v_{j,i} = a_{11}\dot{\sigma}^{2}_{r\theta}+2a_{12}\dot{\sigma}_{r\theta}\dot{\sigma}_{\theta z}+a_{22}\dot{\sigma}^{2}_{\theta z} \tag{36}$$

in which

$$\begin{aligned} a_{11} &= \frac{1+\nu}{E}+\alpha\frac{2}{h}\sigma^{2}_{r\theta} \\ a_{12} &= \alpha\frac{2}{h}\sigma_{r\theta}\sigma_{\theta z} \\ a_{22} &= \frac{1+\nu}{E}+\alpha\frac{2}{h}\sigma^{2}_{\theta z}. \end{aligned} \tag{37}$$

The variational theorem states that I is stationary for the solution to the above rate problem. Hence, by (independently) choosing suitable expressions for v, $\dot{\sigma}_{r\theta}$ and $\dot{\sigma}_{\theta z}$ the method of Ritz can be applied to furnish an approximate solution for these rate or incremental quantities; and, the entire history of stress and deformation can be obtained by solving this rate problem at successive increments of time.

6. LAGRANGIAN FORMULATIONS OF THE THEOREM

Eulerian variables were employed in the foregoing formulation of the rate problem and corresponding variational principle because constitutive equations in plasticity are generally expressed in terms of these variables. However, it is often preferable in solid mechanics to express the relevant field equations using Lagrangian variables, since the initial (undeformed) configuration of the continuum is employed as a reference in the Lagrangian description [13, 14]. This can be achieved by formulating the constitutive law in terms of Lagrangian variables instead of Eulerian variables, as was done in [19, 20].

Using this approach the field equations corresponding to the rate boundary-value problem (17)–(20) become:
the rate of equilibrium in the form

$$\dot{s}_0^{ij}|_i + \rho_0 \dot{f}_0^j = 0 \quad \text{in } V_0 \tag{38}$$

a constitutive law, *postulated* as

$$v_{j|i} = \frac{\partial U(\dot{s})}{\partial \dot{s}_0^{ij}} \quad \text{or } \dot{s}_0^{ij} = \frac{\partial U(v)}{\partial (v_{j|i})} \quad \text{in } V_0 \tag{39}$$

and the following boundary conditions

$$v_j = v_j^* \quad \text{on } S_v^0 \tag{40}$$

$$\dot{F}_0^j = n_i^0 \dot{s}_0^{ij} = \dot{F}_0^{j*} \quad \text{on } S_F^0. \tag{41}$$

In these equations an index 0 implies that the particular quantity is with reference to an element of the continuum in its original configuration. The vertical bar denotes covariant differentiation with respect to the initial coordinates a^i; s_0^{ij} refers to the (unsymmetrical) Lagrangian stress tensor, and the material derivative in this description is obtained by partial differentiation with respect to time. It is postulated that $U(v)$ be a homogeneous potential function of degree two in the velocity gradients $v_{j|i}$ depending on the current state of stress and possibly also on the entire strain history; such that the potential $U(\dot{s})$ can be expressed in terms of Lagrangian stress-rate components through the Legendre transformation

$$U(\dot{s}) + U(v) = \dot{s}_0^{ij} v_{j|i}. \tag{42}$$

A variational formulation for these rate equations involving Lagrangian variables, analogous to (21) becomes

$$I^0 = \int_{V_0} [\dot{s}_0^{ij} v_{j|i} - \rho_0 \dot{f}_0^j v_j - U(\dot{s})] \, dV_0 - \int_{S_F^0} \dot{F}_0^{j*} v_j \, dS^0 - \int_{S_v^0} \dot{F}_0^j (v_j - v_j^*) \, dS^0 \tag{43}$$

and the solution of the rate problem is such that I^0 is stationary in the class of continuous Lagrangian stress-rate and velocity fields. That is, the variational equation

$$\delta I^0 = 0 \tag{44}$$

renders the system of differential equations (38)–(41).

As indicated above, the Lagrangian stress tensor is non-symmetric. For practical applications, to shell problems for example, it is perhaps more convenient to employ relations which are expressed in terms of symmetric tensors. In the Lagrangian description the appropriate variables are the Kirchhoff stress tensor τ^{ij} and Green's strain tensor E_{ij} given respectively by [13, 14]

$$s_0^{ij} = \tau^{ij} + \tau^{ik} u^j_{|k} \tag{45}$$

$$E_{ij} = \tfrac{1}{2}[u_{i|j} + u_{j|i} + u_{k|i} u^k_{|j}] \tag{46}$$

in which u_i are the components of the displacement vector (from the initial configuration). As a result of these relations the following rate quantities are readily obtained

$$\dot{s}_0^{ij} = \dot{\tau}^{ij} + \dot{\tau}^{ik} u^j_{|k} + \tau^{ik} v^j_{|k} \tag{47}$$

$$\dot{E}^{ij} = \tfrac{1}{2}[v_{i|j} + v_{j|i} + v_{k|i} u^k_{|j} + u_{k|i} v^k_{|j}]. \tag{48}$$

Using these variables, the equations of equilibrium (2) and surface tractions can be expressed as [13, 14]

$$(\tau^{ij} + \tau^{ik} u^j_{|k})_{|i} + \rho_0 f_0^j = 0 \tag{49}$$

$$F_0^j = n_i^0(\tau^{ij} + \tau^{ik} u^j_{|k}). \tag{50}$$

The rate equations thus become

$$(\dot{\tau}^{ij} + \dot{\tau}^{ik} u^j_{|k} + \tau^{ik} v^j_{|k})_{|i} + \rho_0 \dot{f}_0^j = 0 \tag{51}$$

$$\dot{F}_0^j = n_i^0(\dot{\tau}^{ij} + \dot{\tau}^{ik} u^j_{|k} + \tau^{ik} v^j_{|k}) \tag{52}$$

which follow simply from (49) and (50) or alternatively, can be derived from a substitution of (47) into (38) and (41).

From (47) and (48), the following relationship is obtained

$$\dot{s}^{ij} v_{j|i} = \dot{\tau}^{ij} \dot{E}_{ij} + \tau^{ij} v^k_{|i} v_{k|j}. \tag{53}$$

Consequently, it can be shown that the constitutive law (39) can also be expressed as follows

$$\dot{E}_{ij} = \frac{\partial W(\dot{\tau})}{\partial \dot{\tau}^{ij}} \quad \text{or } \dot{\tau}^{ij} = \frac{\partial W(v)}{\partial \dot{E}_{ij}} \tag{54}$$

where

$$W = \tfrac{1}{2} \dot{\tau}^{ij} \dot{E}_{ij} \tag{55a}$$

$$= U - \tfrac{1}{2} \tau^{ij} v^k_{|i} v_{k|j} \tag{55b}$$

and

$$W(v) + W(\dot{\tau}) = \dot{\tau}^{ij} \dot{E}^{ij}. \tag{56}$$

That is, if the Lagrangian stress rate $\dot{s}_0^{ij}$ and velocity gradients $v_{j|i}$ are derivable from (rate) potential functions, then Kirchhoff's stress rate $\dot{\tau}^{ij}$ and Green's strain rate $\dot{E}_{ij}$ are also derivable from rate potentials.

In view of (53) and (55b) the functional (43) becomes, in the absence of body force

$$I^0 = \int_{V_0} [\dot{\tau}^{ij} \dot{E}_{ij} + \tfrac{1}{2} \tau^{ij} v^k_{|i} v_{k|j} - W(\dot{\tau})] \, dV_0 - \int_{S_F^0} \dot{F}_0^{j*} v_j \, dS^0 - \int_{S_v^0} \dot{F}_0^j (v_j - v_j^*) \, dS^0. \tag{57}$$

Stationarity of this functional for arbitrary variations of v_j and $\dot{\tau}^{ij}$ produces the rate equations of equilibrium (51) and constitutive law (54) as Euler equations, whereas the boundary conditions of the rate problem (52) and (40) become the natural boundary conditions of the variational equation (44).

If admissible velocity fields are restricted to satisfy the prescribed boundary conditions (40), then from (56) and (57)

$$I^0 \equiv J^0(v) = \int_{V_0} [W(v)+\tfrac{1}{2}\tau^{ij}v^k_{|i}v_{k|j}]\,\mathrm{d}V_0 - \int_{S^0_F} \dot{F}^{j*}_0 v_j\,\mathrm{d}S^0 \tag{58}$$

which is identical to the functional employed in a recent extremum principle developed by Murphy and Lee [20].

For statically admissible stress-rate and velocity distributions, i.e. both satisfying rate equilibrium (51) in V_0 and prescribed boundary conditions on S^0_F, equation (57) and a transformation analogous to (23) gives the complementary to (58):

$$I^0 \equiv G^0(\dot{\tau}, v) = \int_{S^0_v} \dot{F}^j_0 v^*_j\,\mathrm{d}S^0 - \int_{V_0} [\tfrac{1}{2}\tau^{ij}v^k_{|i}v_{k|j} + W(\dot{\tau})]\,\mathrm{d}V_0. \tag{59}$$

Murphy and Lee's extremum principle indicates that any velocity field satisfying (40) furnishes an upper bound to the value of I^0 for the actual solution. However, in view of the coupling between $\dot{\tau}^{ij}$ and v_j in equations (51) and (52), these stress rates and velocity fields cannot be selected independently in the construction of lower bounds G^0. Nevertheless, the stationarity principle (44) in conjunction with the expression (57) permits independent selection of $\dot{\tau}^{ij}$ and v_j in the approximate solution of given elastic–plastic boundary-value problems involving finite deformations. In addition, this general variational principle forms a basis which provides in a consistent manner the relevant field equations for finite deformations of elastic–plastic shells, or other approximations to the three-dimensional theory.

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Абстракт—Рассматривается упругопластическое твердое тело, упрочненное напряжением в квазистатическом состоянии конечной деформации. Развили новый, общий вариационный принцип вопроса граничного значения в котором на однои части граничной поверхности тела предполагается прирост сдвига а на остальной—прирост номинального тягового усилия. Даются кинетические потенциалы как Эйлера, так и Лагранжа для этого вариационного принципа. Нашли, что некоторые принципы экстремум приписываемые Хиллу [11, 12], Мурфи и Ли [20] можно считать специальными случаями недавно порожденной варицаионной теоремы. Также нашли, что этот вариационный принцип можно применять для получения приблизительного разрешения вопроса скручивания эластопластических прутков круглого сечения.